Math 304 (Spring 2015) - Homework 8

Problem 1.

Determine whether the following sets of vectors form an orthonormal basis of $\mathbb{R}^2$.

(a) $\{(1,0)^T, (0,1)^T\}$

(b) $\left\{\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T, \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)^T\right\}$

(c) $\left\{\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}, \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}\right\}$

Solution:

(a) Yes.

(b) Yes.

(c) Yes.

Problem 2.

Let $\{u_1, u_2, u_3\}$ be an orthonormal basis for an inner product space V and let

$$w = u_1 + 2u_2 + 2u_3 \quad \text{and} \quad v = u_1 + 7u_3$$

Determine the value of each of the following:

(a) $\langle w, v\rangle$

(b) $\|w\|$ and $\|v\|$

(c) the angel between w and v.

Solution:

(a) $\langle w, v\rangle = 15$

(b) $\|w\| = 3$ and $\|v\| = \sqrt{50} = 5\sqrt{2}$.

(c)

$$\cos\theta = \frac{\langle w, v\rangle}{\|w\|\|v\|} = \frac{1}{\sqrt{2}}$$

So the angle between w and v is $\pi/4$.

Problem 3.

Given the basis $\{(1,2,-2)^T,(4,3,2)^T,(1,2,1)^T\}$ for $\mathbb{R}^3$, use the Gram-Schmidt process to obtain an orthonormal basis.

Solution: Let us denote

$$v_1 = (1,2,-2)^T, v_2 = (4,3,2)^T, v_3 = (1,2,1)^T.$$

$u_1 = \frac{v_1}{\|v_1\|} = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3})^T$.

Compute $v_2 - \langle v_2, u_1\rangle u_1 = \left(\frac{10}{3}, \frac{5}{3}, \frac{10}{3}\right)^T$, rescale to make it length one:

$$u_2 = (\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3})^T$$

Compute $v_3 - \langle v_3, u_1\rangle u_1 - \langle v_3, u_1\rangle u_1$, then rescale to make it length one. We get

$$u_3 = (-\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3})^T$$

Problem 4.

Let

$$A = \begin{pmatrix} 3 & -1 \\ 4 & 2 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 \\ 20 \\ 10 \end{pmatrix}.$$

(a) Find an orthonormal basis of the column space of A.

(b) Find the projection of v onto the column space of A.

Solution:

(a) An orthonormal basis of A is

$$u_1 = \left(\tfrac{3}{5}, \tfrac{4}{5}, 0\right)^T, u_2 = \left(-\frac{4}{5\sqrt{2}}, \frac{3}{5\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T$$

(b) The projection of v onto the column space of A is

$$\langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2 = (\tfrac{4}{5}, \tfrac{97}{5}, 11)^T$$

Problem 5.

(**Legendre Polynomials**) Let $\mathbb{P}_2 = \{$all polynomials of degree $\leq 2\}$. We define the following inner product on $\mathbb{P}_2$:

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

Start with a basis $\{1, x, x^2\}$ of $\mathbb{P}_2$, use the Gram-Schmidt process to obtain an orthonormal basis.

Solution:

$$\langle 1, 1 \rangle = 2$$

so $u_1 = \frac{1}{\sqrt{2}}$.

$$x - \langle x, u_1 \rangle u_1 = x - 0 = x$$

and

$$\langle x, x \rangle = \sqrt{\tfrac{2}{3}}.$$

so $u_2 = \sqrt{\frac{3}{2}}\, x$.

$$x^2 - \langle x^2, u_1 \rangle u_1 - \langle x^2, u_2 \rangle u_2 = x^2 - \frac{1}{3}$$

so $u_3 = \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})$.